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Quasi-Uniqueness in L^∞ Extremal Problems

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1. INTRODUCTION

Let $H_p{}^k = H_p{}^k[a, b]$ denote the space of functions which are k-fold integrals of functions in $L_p[a, b]$, $1 \leqslant p \leqslant \infty$, and $k = 1, 2, \ldots$. Further, let L be a nonsingular kth-order differential operator with sufficiently smooth coefficients. In this paper, we consider certain seminorm minimization problems of the form

$$\inf\{\| Lf \|_p : f \in F\}, \tag{1.1}$$

where F is a finite codimensional flat in $H_p{}^k$. In Section 2, we are mainly concerned with the case where $p = \infty$ and the F are weak* closed flats given by Hermite–Birkoff interpolation conditions.

The basic idea throughout these sections is to determine to what extent there is uniqueness in the solution to (1.1). This idea was first pursued by Fisher and Jerome in [6]. Of course, there are usually many solutions to (1.1) for $p = \infty$, but we show that under rather general conditions there is an interval on which all solutions differ by at most an element of N_L, the null-space of L. If we make stronger assumptions on the interpolation functionals, we show that there is an interval on which all solutions agree. One important feature of these results is that it is unnecessary to make assumptions regarding the structure of the null-space space of L^* as was done in [6].

In Section 3, we briefly discuss the Favard solution to the L_∞ minimization problem. Theorems analogous to those derived in [1, 2, 3] are stated.

If we assume that

$$F = \{f \in H_\infty{}^k[a, b]: \lambda_i f = \gamma_i,\ i = 1, \ldots, n + k\}$$

then it is interesting to note that the λ's, $\lambda \in \operatorname{span}[\lambda_1, \ldots, \lambda_{n+k}]$, which annihilate N_L generate functions $f(\xi) = \lambda G(\cdot, \xi)$, where $G(x, \xi)$ is the Green's function for L, whose support properties determine the size of the core interval of uniqueness. In fact, in the most common case, when $L = D^k$, one obtains splines which are supported on the convex hull of the support of the functional λ.

The general $H_p{}^k$ minimization problem has been studied by many authors. The interested reader is referred to [7, 8, 12, 13] and the references therein.

We would like to thank Professor Carl de Boor for suggesting these problems to us and for his many helpful constructive criticisms and comments.

2. Essential Uniqueness on Subintervals

Let L be a nonsingular kth-order differential operator on a compact interval $[a, b]$ given by

$$L = D^k + \sum_{j=0}^{k-1} c_j(t)\, D^j \tag{2.1}$$

with $c_j \in C^j[a, b]$, $0 \leqslant j \leqslant k-1$. Here and throughout the paper, $D^j = d^j/dt^j$. Setting $c_k \equiv 1$, the formal adjoint L^* of L is defined by

$$L^*f \equiv \sum_{j=0}^{k} (-1)^j D^j(c_j f).$$

Let N_L and N_{L^*} denote the null-spaces of L and L^*, respectively, and let $G(x, \xi)$ denote the one-sided Green's function of L, namely,

$$\begin{aligned} G(x, \xi) &= g(x, \xi) \qquad \text{if} \quad a \leqslant \xi \leqslant x, \\ &= 0 \qquad\qquad \text{if} \quad x < \xi \leqslant b, \end{aligned} \tag{2.2}$$

with $L_x G(x, \xi) = \delta_\xi$, the delta distribution at ξ, where the subscript x indicates that the differential operator L is applied to G with respect to the variable x. It is well known that we can write

$$g(x, \xi) = \sum_{i=1}^{k} \phi_i(x)\, \phi_i^*(\xi), \tag{2.3}$$

where $\{\phi_1, \ldots, \phi_k\}$ and $\{\phi_1^*, \ldots, \phi_k^*\}$ are bases of N_L and N_{L^*}, respectively. Thus, for any $u \in L^\infty[a, b]$,

$$z(x) = \int_a^b G(x, \xi)\, u(\xi)\, d\xi \tag{2.4}$$

satisfies (a.e.) $Lz = u$ with $z(a) = z'(a) = \cdots = z^{(k-1)}(a) = 0$. If $L = D^k$, one can easily check that

$$G(x, \xi) = (x - \xi)_+^{k-1}/(k-1)!.$$

Let $f_0 \in H_\infty{}^k[a, b]$ and Λ be a finite subset of $(H_\infty{}^k[a, b])^*$. We will be concerned with the seminorm minimization problem

$$\inf\{\| Lf \|_\infty : f \in H_\infty{}^k[a, b],\ \lambda f = \lambda f_0 \text{ for all } \lambda \in \Lambda\}. \tag{2.5}$$

Now, $H_\infty{}^k$ is clearly the dual of $H_1{}^k$. It is well known that if all the $\lambda \in \Lambda$ are weak* continuous then (2.5) has a solution (cf. [11]).

In this paper we will only consider the case where the support of each linear functional λ consists of a single point; that is,

$$\lambda(f) \equiv \lambda_{jr}(f) = \sum_{i=0}^{k-1} \alpha_{j,i} f^{(i)}(x_r), \qquad x_r \in [a, b]. \tag{2.6}$$

We will assume that $p < q$ implies $x_p \leqslant x_q$ and that $\{x_1, \ldots, x_m\} = \bigcup_{\lambda \in \Lambda} \operatorname{supp} \lambda$.

Let $M = \operatorname{span} \Lambda$ and

$$N = \{\mu_x G(x, \cdot) : \mu \in M \cap (N_L)^\perp\}.$$

Then (2.5) is equivalent to

$$\inf \left\{ \| g \|_\infty : \int_a^b ng = \int_a^b nLf_0 \text{ for all } n \in N \right\} \tag{2.7}$$

in the sense that f_* solves (2.5) if and only if $g_* = Lf_*$ solves (2.7). This equivalency is discussed in detail in [4], where M is a finite-dimensional subspace generated by Hermite interpolation data and $L = D^k$.

The following result is the key lemma to all further results in this paper.

LEMMA 2.1. *Let $f \in N$ and suppose that f vanishes on $(x_r - \epsilon, x_r)$ and $(x_s, x_s + \epsilon)$ for some $r < s$ and $\epsilon > 0$. If $\{\lambda \in \Lambda : \operatorname{supp}(\lambda) \subset [x_r, x_s]\}$ is linearly independent over N_L, then f vanishes also on $[x_r, x_s]$.*

Proof. $f = \mu_x G(x, \cdot)$, where $\mu = \sum_{\lambda \in \Lambda} a_\lambda \lambda$. Thus,

$$\begin{aligned} f(\xi) &= \sum_{\operatorname{supp}\lambda \geqslant \xi} a_\lambda \lambda_x G(x, \xi) \\ &= \sum_{j=1}^{k} \phi_j{}^*(\xi) \sum_{\operatorname{supp}\lambda \geqslant \xi} a_\lambda \lambda \phi_j . \end{aligned} \tag{2.8}$$

Since the $\{\phi_j^*\}_{j=1}^k$ are linearly independent over any open interval and f vanishes on $(x_r - \epsilon, x_r)$ and $(x_s, x_s + \epsilon)$ we conclude that

$$\begin{aligned} &\text{(i)} \quad \sum_{\operatorname{supp}\lambda \geqslant x_r} a_\lambda \lambda = 0 \quad \text{on} \quad N_L, \\ &\text{(ii)} \quad \sum_{\operatorname{supp}\lambda > x_s} a_\lambda \lambda = 0 \quad \text{on} \quad N_L. \end{aligned} \tag{2.9}$$

Thus, $\sum_{x_r \leqslant \operatorname{supp}\lambda \leqslant x_s} a_\lambda \lambda = 0$ on N_L and, hence, by linear independence, $a_\lambda = 0$ for λ satisfying $x_r \leqslant \operatorname{supp} \lambda \leqslant x_s$. It is now easy to see that for $x_r \leqslant \xi \leqslant x_s$

$$f(\xi) = \sum_{j=1}^{k} \phi_j^*(\xi) \sum_{\operatorname{supp}\lambda > x_s} a_\lambda \lambda \phi_j = 0$$

by (2.9, ii), and this completes the proof.

In order to state our main result we need introduce one more notation. Set

$$\begin{aligned} \Lambda_{ij} &\equiv \{\lambda \in \Lambda \colon \operatorname{supp} \lambda \subset [x_i, x_j]\}, \\ i' &\equiv \min\{j \geqslant i \colon \Lambda_{ij} \text{ is linearly dependent on } N_L\}. \end{aligned} \tag{2.10}$$

THEOREM 2.1. *Let* Λ and $\{x_1, ..., x_m\}$ *be as above. Then the minimization problem* (2.5) *has a solution. Furthermore, there is an interval J on which all solutions of* (2.5) *differ by not more than elements of N_L and such that, for some i, J contains the interval* $[x_i, x_{i'}]$ *with i' defined in* (2.10).

Proof. By the Duality Theorem, there is a function $n_* \in N$ with $\| n_* \|_1 = 1$ so that every solution g_* of (2.7) must satisfy

$$\| g_* \|_\infty = \int_a^b n_* g_* . \tag{2.11}$$

Furthermore, $g_* = \| g_* \|_\infty \operatorname{sgn} n_*$ on the support of n_*. Since $n_* \in N$ is not identically zero, Lemma 2.1 implies that there must be an interval $[x_i, x_{i'}]$ contained in the support of n_*. This completes the proof of the theorem.

Several remarks are now in order. Note first that if we assume that Λ is linearly independent then $i' \geqslant i + 1$. In [6], the number k_0 is defined by

$$k_0 \equiv \max\{j \colon \operatorname{card} \Lambda_{i+1, i+j} \leqslant k, \text{ all } i\}.$$

This number k_0 measures the largest possible number of consecutive points which could support linearly independent elements of Λ over N_L. We may now state

COROLLARY 2.1. *If $\Lambda_{i+1,i+k_0}$ is linearly independent over N_L for $i = 1,\ldots, m - k_0$ then $i' \geqslant i + k_0$ and hence all solutions of* (2.5) *differ by no more than elements of N_L on some interval of the form $[x_i, x_{i'}]$.*

This result is "closest" to Theorem 2 of Fisher–Jerome in [6], although their theorem is misstated since they claim that there are at least $k + 1$ elements of Λ supported on the interval $[x_i, x_{i'}]$ given in Corollary 2.1, but their hypotheses are not strong enough to guarantee this. However, one can obtain this conclusion with a stronger hypotheses as follows:

COROLLARY 2.2. *If* card $\Lambda_{ij} \leqslant k$ *implies that Λ_{ij} is linearly independent over N_L for all meaniningful i and j, then all solutions of* (2.5) *differ by no more than elements of N_L on some interval of the form $[x_i, x_{i'}]$ where* card $\Lambda_{i,i'} > k$.

We have not yet been able to conclude that all solutions are equal on some core interval. We now see that if one makes essentially the strongest hypotheses then one can obtain a result on uniqueness.

THEOREM 2.2. *If for all meaningful i and j* card $\Lambda_{ij} \leqslant k$ *implies that Λ_{ij} is linearly independent over N_L and* card $\Lambda_{ij} > k$ *implies that Λ_{ij} is total over N_L then there is an interval of the form $[x_i, x_i']$ so that* card $\Lambda_{ii'} > k$ *and all solutions of* (2.5) *are equal on this interval.*

The proof of this theorem is just a quick application of Corollary 2.2. One merely notes that the totality of $\Lambda_{ii'}$ forces all solutions to be equal on $[x_i, x_{i'}]$.

3. FAVARD'S SOLUTION

In this section, we discuss a method for singling out a unique solution, called Favard's solution, to the L_∞-minimization problem (2.5) when $L \equiv D^k$. Favard proposed this method in [5]. Recently, de Boor [1] was able to interpret Favard's remarks into a viable algorithm for producing this solution as well as proving unicity for $L \equiv D^k$. In [2, 3] the authors showed that solutions to L^p minimization problems corresponding to (2.5) converge weak* as $p \to \infty$ to Favard's solution. It follows that Favard's solution can be seen to be the (unique) limit of a Pólya-type algorithm [10, p. 246] for solving the L_∞ minimization problem.

Favard's solution is obtained by solving a finite sequence of L_∞ minimization problems on nested domains (cf. [1] for more details). It follows that Favard's solution and the strict approximation of Rice [10, p. 239] are essentially the same since they can both be determined by a sequence of finite dimensional dual problems. Let

$$G_p^k = \{f \in H_p^k[a, b]: \lambda_i f = \lambda_i f_0 , i = 1,..., n + k\}.$$

If the $\{\lambda_i\}_{i=1}^{n+k}$ are total over N_L it is easily seen that the seminorm minimization problem

$$\inf\{\| Lf \|_p : f \in G_p^k\} \tag{3.1}$$

has a unique solution $S_p \in G_p^k$ when $1 < p < \infty$. We may now state a a theorem which is an immediate consequence of [1, 3].

THEOREM 3.1. *Suppose that the $\{\lambda_i\}_{i=1}^{n+k}$ are as in (2.6) and that the $\{\lambda_i\}_{i=1}^{n+k}$ are total over N_L. Then the Favard solution, which we denote by S_∞, is unique and*

$$S_p \to S_\infty \tag{3.2}$$

as $p \to \infty$ where the convergence is $w^*(H_\infty{}^k)$.

We remark that this theorem remains true if we allow the functionals λ_i to be in $(C^{k-1}[a, b])^*$.

Certain structural results concerning the Favard solution S_∞ may be derived if more information concerning the differential operator L and/or the linear functionals $\{\lambda_i\}_{i=1}^{n+k}$ is available. We present here one of the strongest results available based on the assumption that L is totally disconjugate and the $\{\lambda_i\}_{i=1}^{n+k}$ are Hermite interpolation functionals. We say that L is totally disconjugate [9, p. 501] if

$$L = D_k D_{k-1} \cdots D_1 ,$$

where $D_i f = D((1/w_i) f)$ and $w_i > 0$ with $w_i \in C^k[a, b]$. An exhaustive study of operators of this type may be found in [9]. In particular, if the $\{\lambda_i\}_{i=1}^{n+k}$ are represented by Hermite interpolation, then we have

THEOREM 3.2. *If L is totally disconjugate and the $\{\lambda_i\}_{i=1}^{n+k}$ represent Hermite interpolation data of order less than k at the points $\{x_i\}_{i=1}^m$, then the Favard solution S_∞ to (2.5) is unique and satisfies*

(i) $(LS_\infty)(t) = 0, \qquad t \notin [x_1 , x_m],$

(ii) $| LS_\infty |$ *is piecewise constant and has discontinuitites only at* $\{x_i\}_{i=1}^m$,

(iii) LS_∞ *has fewer than n jumps in* (x_1 , x_m).

The proof of this theorem is essentially contained in [1]. It relies heavily on the variation-diminishing properties of the Basic Splines studied by Karlin [9, p. 522].

One can also ask what happens to S_p as $p \to 1$. The authors have considered this problem in [4] when $L \equiv D^k$ and have obtained partial results on the convergence of subsequences in $(NBV)^k$.

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